Quotient rule and chain-rule.

Q1. Compute the derivative of x^{-n} and x^{α} (n is positive integer and α is real number)

Quotient rule:

$$(x^{-n})' = \left(\frac{1}{x^n}\right)' = \frac{(1)'x^n - 1\cdot(x^n)'}{x^{2n}} = \frac{0\cdot x^n - nx^{n-1}}{x^{2n}} = -nx^{-(n+1)}.$$

Chain-rule:

$$x^{\alpha} = e^{\ln x^{\alpha}} = e^{\alpha \ln x}$$

set $f(x) = e^x$, $g(x) = \alpha \ln x$, so $x^{\alpha} = f\circ g(x)$

$$(x^{\alpha})' = f'(g(x))\, g'(x) = e^{\alpha\ln x}\cdot \alpha\frac{1}{x} = x^{\alpha}\cdot\alpha\cdot x^{-1} = \alpha x^{\alpha-1}$$

Q2. (1) $f(x) = \ln(x+\sqrt{1+x^2})$; (2) $f(x) = \tan^2\frac{1}{x}$.

(1) $f(x) = \ln(x+\sqrt{1+x^2})$

set $g(x) = \ln x$, $h(x) = x+\sqrt{1+x^2}$

$$f'(x) = (g\circ h(x))' = g'(h(x))\cdot h'(x) = \frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{1}{2}\frac{2x}{\sqrt{1+x^2}}\right) = \frac{1}{\sqrt{1+x^2}}$$

(2) $f(x) = \tan^2\frac{1}{x}$

set $g(x) = x^2$, $h(x) = \tan x$, $m(x) = \frac{1}{x}$,

$$f'(x) = (g\circ h\circ m(x))' = g'(h(m(x)))\, h'(m(x))\, m'(x)$$

$$= 2\left(\tan\frac{1}{x}\right)\cdot(\tan x)'\left(-\frac{1}{x^2}\right) = -2\tan\frac{1}{x}\sec^2\frac{1}{x}\cdot\frac{1}{x^2}$$

for $\tan x$, we use quotient rule:

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x\cdot\cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

similarly we have
$$\begin{cases}(\sec x)' = \sec x\tan x\\ (\cot x)' = -\frac{1}{\sin^2 x} = -\csc^2 x\\ (\csc x)' = -\csc x\cot x.\end{cases}$$

Q3. (1) $f(x)=g(x+g(x))$; (2) $f(x)=g(xg(x))$

(1) set $h(x)=x+g(x)$

$$f'(x)=(g\circ h(x))'=g'(x+g(x))\cdot\underbrace{(1+g'(x))}_{=h'(x)}$$

(2) set $h(x)=xg(x)$

$$f'(x)=(g\circ h(x))'=g'(xg(x))\,\underbrace{(g(x)+xg'(x))}_{=h'(x)}$$

A concrete example may like set $g(x)=\sin x$.

so for (1) $f(x)=\sin(x+\sin x)$

then $f'(x)=\cos(x+\sin x)\cdot(1+\cos x)$

for (2) $f(x)=\sin(x\sin x)$

$f'(x)=\cos(x\sin x)\cdot(\sin x+x\cdot\cos x)$

Q4. $y=\dfrac{(x+5)^2(x-4)^{\frac{1}{3}}}{(x+2)^5(x+4)^{\frac{1}{2}}}$ $(x>4)$, compute y'

For such complex function, we need use the "ln" function to make it simpler. like:

$$\ln y=\ln\frac{(x+5)^2(x-4)^{\frac{1}{3}}}{(x+2)^5(x+4)^{\frac{1}{2}}}=\ln(x+5)^2(x-4)^{\frac{1}{3}}-\ln(x+2)^5(x+4)^{\frac{1}{2}}$$

$$=2\ln(x+5)+\frac{1}{3}\ln(x-4)-5\ln(x+2)-\frac{1}{2}\ln(x+4)$$

then do the differential both sides.

$(LHS)'=(\ln y)'=\frac{1}{y}\cdot y'$ (chain-rule)

$(RHS)'=\frac{2}{x+5}+\frac{1}{3}\frac{1}{x-4}-\frac{5}{x+2}-\frac{1}{2(x+4)}$ ①

so $y'=y\cdot$ ① where $y=\dfrac{(x+5)^2(x-4)^{\frac{1}{3}}}{(x+2)^5(x+4)^{\frac{1}{2}}}$

Q5. For the general form $u(x)^{v(x)}$, also use "ln" function. and chain-rule.

$$u(x)^{v(x)} = e^{\ln u(x)^{v(x)}} = e^{v(x)\ln u(x)}$$

$$(u(x)^{v(x)})' = (e^{v(x)\ln u(x)})' = e^{v(x)\cdot\ln(u(x))}\cdot(v(x)\cdot\ln u(x))'$$

$$= u(x)^{v(x)}\left(v'(x)\ln u(x) + v(x)\cdot\frac{1}{u(x)}u'(x)\right)$$

Q6. (1) $x^{\sin x}$; (2) x^{x^x}

(1) just an application of Q5, $u(x)=x$, $v(x)=\sin x$

$$(x^{\sin(x)})' = x^{\sin x}\left(\cos x\cdot\ln x + \sin x\cdot\frac{1}{x}\right)$$

(2) x^{x^x}, use "ln" function.

$$y = x^{x^x} \Rightarrow \ln y = \ln x^{x^x} = x^x \ln x.$$

still have x^x term, use "ln" once more

$$\ln(\ln y) = \ln x^x + \ln(\ln x) = x\ln x + \ln(\ln x)$$

differential both sides:

$$\frac{1}{\ln y}\cdot\frac{1}{y}\cdot y' = \left(1\cdot\ln x + x\cdot\frac{1}{x}\right) + \frac{1}{\ln x}\cdot\frac{1}{x} = \ln x + 1 + \frac{1}{x\ln x}$$

$$\Rightarrow y' = \ln y\cdot y\left(\ln x + 1 + \frac{1}{x\ln x}\right) = x^x \ln x \cdot x^{x^x}\cdot\left(\ln x + 1 + \frac{1}{x\ln x}\right)$$

Tutorial 5. Min. Jie jmin@math.cuhk.edu.hk

便笺标题 2014/10/6

Compute derivative using

(1) product rule

$$(f(x)\cdot g(x))' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

(2) quotient rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}$$

(3) Chain rule

$$(f(g(x)))' = f'(g(x))\cdot g'(x)$$

Examples: (1) $f(x) = \frac{x^2+1}{x+1}$

use quotient rule:

$$f'(x) = \frac{(x^2+1)'(x+1) - (x^2+1)(x+1)'}{(x+1)^2} = \frac{2x(x+1) - (x^2+1)}{(x+1)^2}$$

$$= \frac{x^2+2x-1}{x^2+2x+1}$$

(2) $f(x) = 3\cdot\sec x - \tan x$

$$f'(x) = 3\cdot\left(\frac{1}{\cos x}\right)' - \left(\frac{\sin x}{\cos x}\right)' \quad \text{(use quotient rule)}$$

$$= 3\cdot\frac{1'\cos x - (-\sin x)}{\cos^2 x} - \frac{\cos^2 x - \sin x(-\sin x)}{\cos^2 x}$$

$$= 3\cdot\frac{\sin x}{\cos^2 x} - \frac{1}{\cos^2 x} = \frac{3\sin x - 1}{\cos^2 x}$$

(3) $f(x) = \ln(\ln x)$

$f'(x) = \ln'(\ln x) \cdot (\ln x)'$ use chain rule

$= \frac{1}{\ln x} \cdot \frac{1}{x}$

(4) $f(x) = 3^x$

If there is x in the power, say $f(x)^{g(x)}$

usually write it as $e^{g(x) \cdot \ln f(x)}$

$f(x) = 3^x = e^{x \ln 3}$

$f'(x) = (e^{x\ln 3})' = e^{x\ln 3} \cdot (x\ln 3)' = e^{x\ln 3} \cdot \ln 3 = 3^x \ln 3$

use chain rule

(5) $f(x) = x^x = e^{x\ln x}$

$f'(x) = (e^{x\ln x})'$ use chain rule

$= e^{x\ln x}(x\ln x)'$ use product rule

$= e^{x\ln x}(x'\ln x + x(\ln x)')$

$= e^{x\ln x}(\ln x + 1)$

$= x^x(\ln x + 1)$

Exercises: compute following derivatives:

(1) $f(x) = \ln(x + \sqrt{1+x^2})$

(2) $f(x) = x^{\sqrt{x}}$

(3) $f(x) = \frac{\tan x}{\sqrt{x}}$

(4) (Final 2005-06) $f(x) = x^n(x-1)^m$, n, m are natural numbers

find $c \in (0,1)$ s.t. $f'(c) = 0$

Sol'n : (1) $f'(x) = \frac{1}{x+\sqrt{1+x^2}} \cdot \left(1 + \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot (2x)\right)$

$$= \frac{1}{x+\sqrt{1+x^2}} \left(1 + \frac{x}{\sqrt{1+x^2}}\right)$$

$$= \frac{1}{x+\sqrt{1+x^2}} \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}$$

(2) $f(x) = x^{\sqrt{x}} = e^{\sqrt{x}\ln x}$

$$f'(x) = e^{\sqrt{x}\ln x} (\sqrt{x}\ln x)' = e^{\sqrt{x}\ln x} \left(\frac{1}{2\sqrt{x}}\ln x + \frac{\sqrt{x}}{x}\right)$$

$$= x^{\sqrt{x}} \frac{\ln x + 2}{2\sqrt{x}}$$

(3) $f'(x) = \frac{(\tan x)'\sqrt{x} - \frac{1}{2\sqrt{x}}\tan x}{x} = \frac{\frac{\sqrt{x}}{\cos^2 x} - \frac{\tan x}{2\sqrt{x}}}{x}$

$$= \frac{1}{\sqrt{x}\cos^2 x} - \frac{\tan x}{2x\sqrt{x}}$$

(4) ① if $n, m \geq 1$, then.

$$f'(x) = n x^{n-1}(x-1)^m + m x^n (x-1)^{m-1}$$

$$= x^{n-1}(x-1)^{m-1}\left(n(x-1) + mx\right) = 0$$

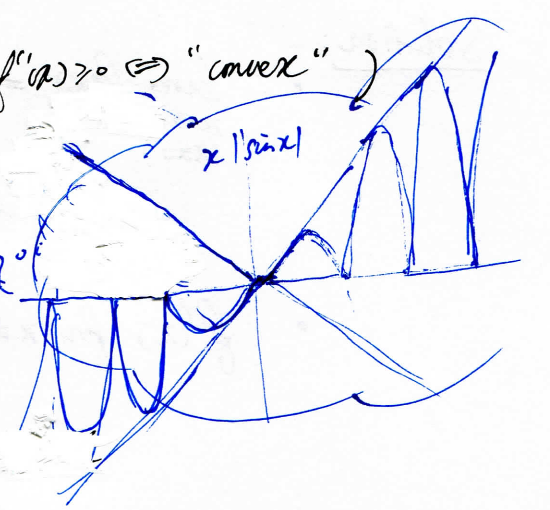
Since $x \neq 0$, $x \neq 1$, so $x = \frac{n}{m+n}$

② if either $n=0$, or $m=0$, then.

there is no sol'n.

[Math 1010 C] 9/10/2014 Tutorial-5 Jiang Qingyuan

- Prof. Yung: $\max\{Q_1, Q_2\}$ instead of Q_1;
- Homework-2;
- Curve sketching; ($f'(x) \geq 0 \Leftrightarrow$ increasing; $f''(x) \geq 0 \Leftrightarrow$ "convex")



Part I: Questions about Homework-2?

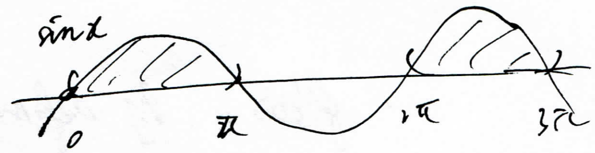
Possible ones:

Assign-2, 5| Find $\frac{dy}{dx}$ if $y = x|\sin x|$;

Solution]: • when $x \in (2k\pi, (2k+1)\pi)$, $k \in \mathbb{Z}$

$$y = x \sin x;$$

$$\frac{dy}{dx} = \sin x + x \cos x;$$



• when $x \in ((2k+1)\pi, 2k\pi)$, $k \in \mathbb{Z}$

$$y = -x \sin x;$$

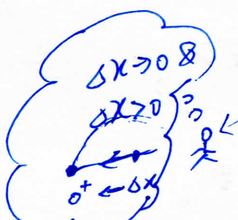
$$\frac{dy}{dx} = -\sin x - x \cos x;$$

What happens when $x = 2k\pi$ or $x = (2k+1)\pi$?

• When $x = 2k\pi$, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ should be computed sep. for $\Delta x > 0$ & $\Delta x < 0$:

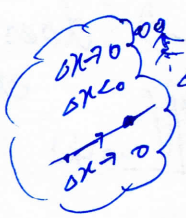
$$\lim_{\Delta x \to 0^+} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{(2k\pi + \Delta x)\overset{(>0)}{\sin(2k\pi + \Delta x)}}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{(2k\pi + \Delta x)\sin \Delta x}{\Delta x}$$

$$= 2k\pi,$$



$$\lim_{\Delta x \to 0^-} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-(2k\pi + \Delta x)\overset{(<0)}{\sin(2k\pi + \Delta x)}}{\Delta x} = -2k\pi;$$

$\neq 2k\pi$ unless $k = 0$



Hence $\left.\frac{dy}{dx}\right|_{x=0} = 0$, $\left.\frac{dy}{dx}\right|_{x=2k\pi}$ does not exist!! for $k \neq 0$, $k \in \mathbb{Z}$.

• When $x = (2k+1)\pi$, similarly,

$$\lim_{\Delta x \to 0^+} \frac{\Delta y}{\Delta x} = -(2k+1)\pi. \quad \lim_{\Delta x \to 0^-} \frac{\Delta y}{\Delta x} = (2k+1)\pi \neq -(2k+1)\pi \quad \forall k \in \mathbb{Z}$$

Hence $\left.\frac{dy}{dx}\right|_{x=(2k+1)\pi}$ does not exist for $\forall k \in \mathbb{Z}$!! □

6] $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$ $f'(x)$ exists $\forall x \in \mathbb{R}$, but not cont. at $x=0$.

Solution:

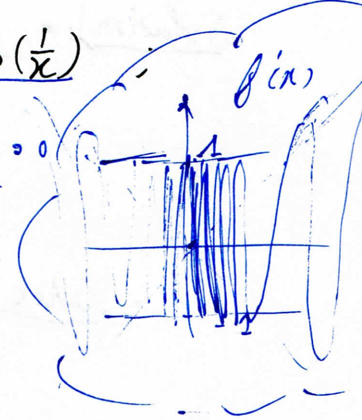
- $\lim\limits_{\substack{x \to 0 \\ (x \neq 0)}} f(x) = \lim\limits_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$, since $\left|\sin\left(\frac{1}{x}\right)\right| \leq 1$,

  $\Rightarrow f(x)$ is cont. at $x=0$;

- $f'(x)$ for $x \neq 0$: $\underline{f'(x)} = \left(x^2 \sin\left(\frac{1}{x}\right)\right)' = 2x \sin\left(\frac{1}{x}\right) + x^2 \cdot \left(-\frac{1}{x^2}\right) \cos\left(\frac{1}{x}\right)$

  $$= \underline{2x \sin\left(\tfrac{1}{x}\right) - \cos\left(\tfrac{1}{x}\right)}$$

  for $x \neq 0$



- $f'(0)$: By definition of derivatives,

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin\left(\frac{1}{x}\right) - 0}{x - 0}$$

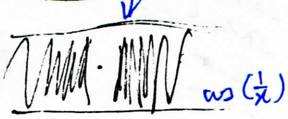
$$= \lim_{x \to 0} x \cdot \sin\left(\tfrac{1}{x}\right) = 0, \quad \text{since } \left|\sin\left(\tfrac{1}{x}\right)\right| \leq 1;$$

$\Rightarrow$ $\underline{f'(0) \text{ exists}}$, & $\underline{f'(0) = 0}$;

Now clearly $f'(x)$ exists for $\forall x \in \mathbb{R}$:

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), & x \neq 0; \\ 0, & x = 0; \end{cases}$$

but $\lim\limits_{\substack{x \to 0 \\ (x \neq 0)}} f'(x) = \lim\limits_{x \to 0} \left(\underbrace{2x \sin\left(\tfrac{1}{x}\right)}_{\downarrow\, 0} - \cos\left(\tfrac{1}{x}\right)\right)$ does not exist!



$\Rightarrow f'(x)$ is not continuous at $x=0$. (!)

Assignment 2 (due date: 3 Oct. (Friday))
From MATH 1010A webpage.
(Inserted Here For Reference Only.)

MATH1010 University Mathematics 2014-2015
Assignment 2
Due: 3 Oct 2013 (Friday)

Answer all questions.

1. Evaluate the following limits.

   (a) $\displaystyle\lim_{x\to 4}\frac{x^2-16}{x^2-2x-8}$

   (b) $\displaystyle\lim_{x\to 3}\frac{\sqrt{x+6}-3}{x^3-27}$

   (c) $\displaystyle\lim_{x\to 4}\frac{8-x^{\frac{3}{2}}}{16-x^2}$

   (d) $\displaystyle\lim_{x\to 0}\frac{1}{x}\left(\frac{1}{\sqrt{1-x}}-\frac{1}{\sqrt{1+x}}\right)$

   (e) $\displaystyle\lim_{x\to 0}\frac{\tan^2 x}{\sin(x^2)}$

   (f) $\displaystyle\lim_{x\to 0}\frac{\sin^2 x}{1-\sqrt{\cos x}}$

2. Let $f(x)$ be a function. Prove that if $\lim_{x\to a}|f(x)|=0$, then $\lim_{x\to a}f(x)=0$.

3. Use definition to evaluate the derivatives of the following functions.

   (a) $y=\dfrac{3}{x^2}$

   (b) $y=2\sqrt{x}-1$

4. Find $\dfrac{dy}{dx}$ if

   (a) $y=x^4\cos 5x$

   (b) $y=\dfrac{e^{-x}}{\sqrt{x}}$

   (c) $y=e^{\sin 3x}$

   (d) $y=\dfrac{x}{\sqrt{x^2+1}}$

   (e) $y=\sec^2 x$

   (f) $y=\ln(2+\sin(x^2+1))$

   (g) $y=\cos\left(\dfrac{1}{\cosh x}\right)$

   (h) $y=\sqrt{\dfrac{1+\sin x}{1-\sin x}}$

   (i) $y=\ln(\ln(x^4+1))$

5. Find $\dfrac{dy}{dx}$ if $y=x|\sin x|$.

6. This exercise shows that the derivative of a function may not be continuous. Let

$$f(x)=\begin{cases} x^2\sin\left(\dfrac{1}{x}\right), & \text{when } x\neq 0\\ 0, & \text{when } x=0\end{cases}$$

   (a) Show that $f(x)$ is continuous at $x=0$.

   (b) Find $f'(x)$ for $x\neq 0$.

   (c) Show that $f(x)$ is differentiable at $x=0$ by evaluating $f'(0)$.

   (d) Explain whether $f'(x)$ is continuous at $x=0$.

End

Assignment 2 (due date: 3 Oct. (Friday))
From MATH 1010A webpage.
(Inserted Here For Reference Only.)

Part II Curve Sketching

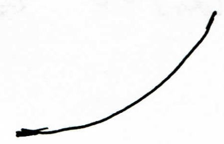
Good Reference: M. Fikhtengol's, "Differential and Integral Calculus", Volume I, chapter 4, §3. (In Russian)

(Chinese version: T.M. 菲赫金哥尔茨, 微积分学教程, (第8版), 高等教育出版社, 第一卷, 第四章 §3. 函数的作图.)

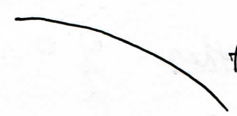
Online access available at CUHK library webpage: just search !!

Rule no.1: $f \in C[a,b]$

- $f'(x) > 0 \Longrightarrow$ increasing



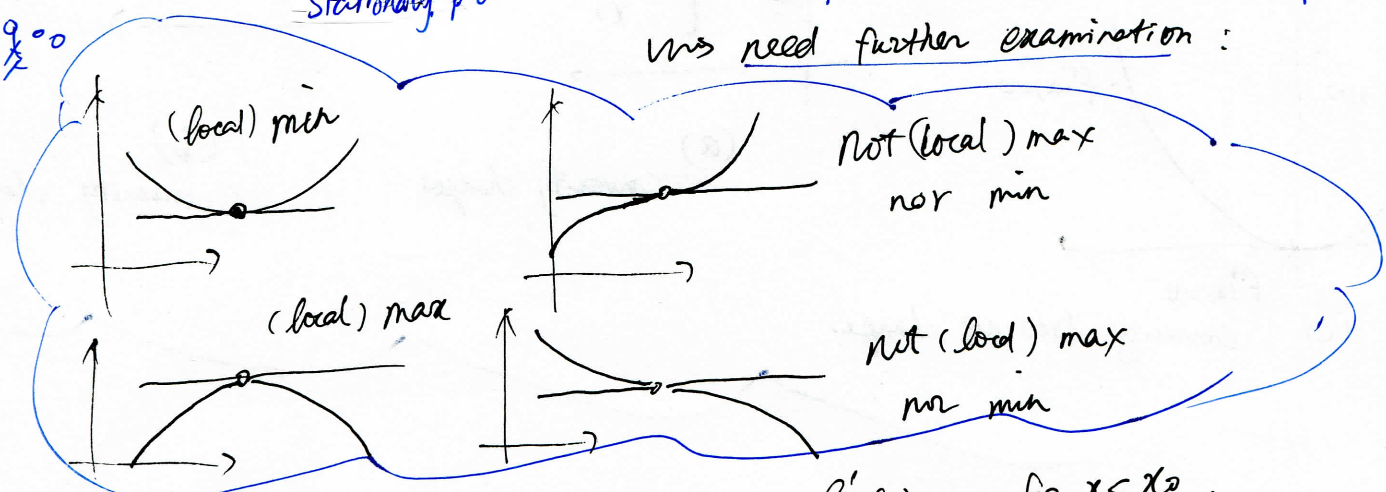
- $f'(x) < 0 \Longrightarrow$ decreasing



- $f'(x_0) = 0 \Longleftarrow$ $x_0 \in (a,b)$ & x_0 is max or min;

"Stationary pt" $\Longrightarrow$ x_0 is potential max or min point.

⇝ need further examination:



Rules (Thm): If $f'(x_0) = 0$, & $f'(x) < 0$ for $x < x_0$, $f'(x) > 0$ for $x > x_0$,

then $x = x_0$ is a (local) min;

If $f'(x_0) = 0$, & $f'(x) > 0$ for $x < x_0$, $f'(x) < 0$ for $x > x_0$,

then $x = x_0$ is a (local) max;

\#

Rules (thm): If $f \in C^2[a,b]$, $x_0 \in (a,b)$.

- $f'(x_0) = 0$, $f''(x_0) > 0 \Rightarrow x = x_0$ loc. min;

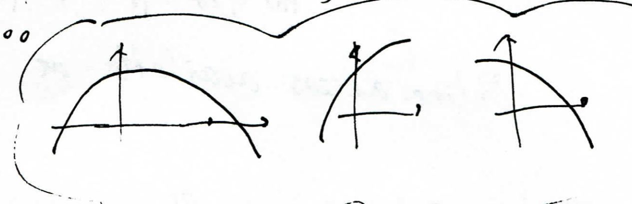
- $f'(x_0) = 0$, $f''(x_0) < 0 \Rightarrow x = x_0$ loc. max;

\#

Rule no. 2: $f''(x)$ (second derivatives; i.e. derivative of $f'(x)$).

- $f''(x) > 0 \implies$ "Convex" (upper convex)

- $f''(x) < 0 \implies$ "Concave" (lower convex)

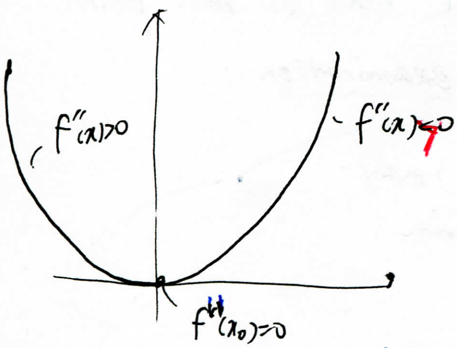


- $f''(x_0) = 0 \iff$ "Inflection point"!

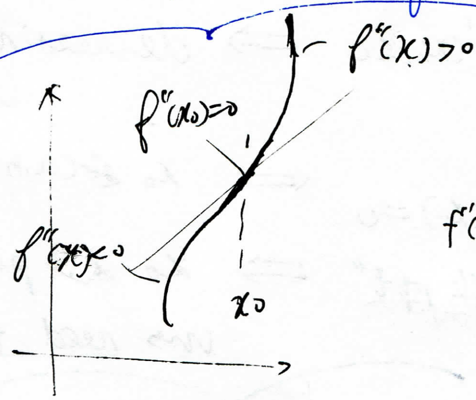
$\longleftrightarrow$ suspect point of where convexity changes:

⇝ need further examination:

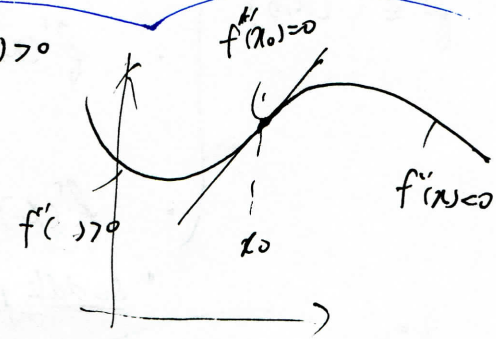
e.g. $f(x) = x^4$



(c) convexity does not change.



(a) convexity changes



(b) convexity changes

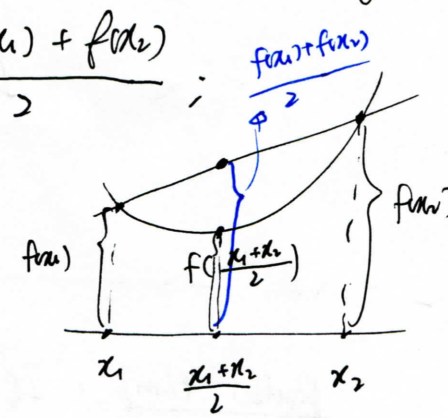
Convex function

Def'n: f continuous f'n on $[a,b]$; f is called convex if

$$\forall\ a \le x_1 < x_2 \le b,\quad f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2};$$

f is called concave if

$$\forall\ a \le x_1 < x_2 \le b,\quad f\left(\frac{x_1+x_2}{2}\right) \ge \frac{f(x_1)+f(x_2)}{2};$$



(Convex function continued). For f is cont. f'n on $[a,b]$. TFAE (The Following Are Equivalent)

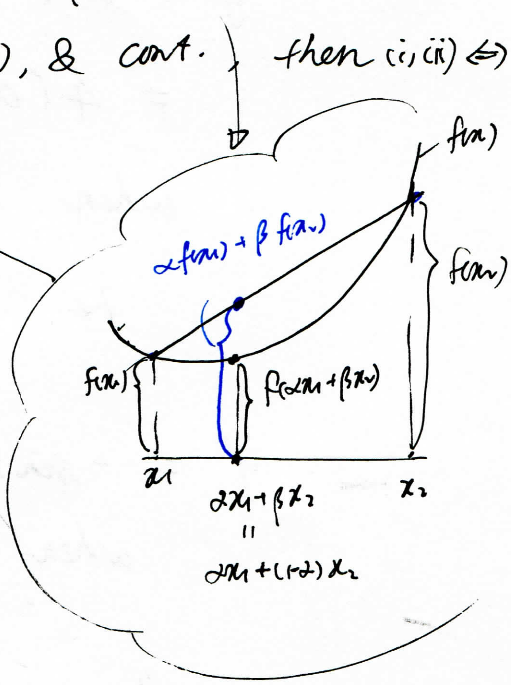
(i) f is convex;

(ii) $\forall\ a \le x_1 < x_2 \le b$, & $\alpha, \beta \in (0,1)$, $\alpha+\beta=1$,

$$f(\alpha x_1 + \beta x_2) \le \alpha f(x_1) + \beta f(x_2);$$

When $f \in C^2[a,b]$, i.e. $f''(x)$ exists $\forall x \in (a,b)$, & cont. then (i) (ii) $\Leftrightarrow$

(iii) $f''(x) \ge 0$, $\forall x \in (a,b)$.



Example-1 $f(x) = x^3 - 3x$;

(· $f(x)=0 \Leftrightarrow x=0$ or $x = \pm\sqrt{3}$;)

· $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$;

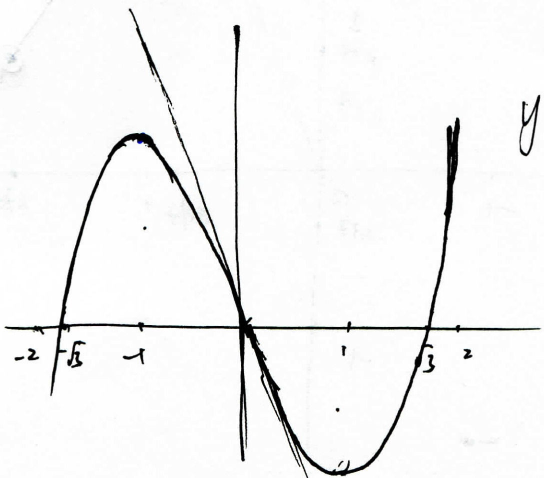
$x=1$, $f(1) = -2$;

$x=-1$, $f(-1) = 2$;

· $f''(x) = 6x$; $f''(x)=0 \Leftrightarrow x=0$;

list all information in a box:

x	$-\infty$		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$		$+\infty$
$f(x)$	$-\infty$		0		2		0		-2		0		$+\infty$
					loc max		inflection		loc min				
$f'(x)$		>0			0		<0 (-3)		0		>0		
$f''(x)$			<0				0		>0				



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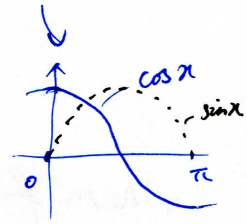
Example 2 (Fikhtengol's [147] 2).) $y = \sin x + \sin 2x$;

Observe: y is periodic, w/ period 2π; & y is odd

$\rightsquigarrow$ need only to sketch in interval $[0, \pi]$.



Now:

$$y' = \cos x + 2\sin 2x = 4\cos^2 x + \cos x - 2$$

$$= 4\left(\cos x + \frac{1+\sqrt{33}}{8}\right)\left(\cos x - \frac{-1+\sqrt{33}}{8}\right)$$

when $\cos x = \dfrac{1 \pm \sqrt{33}}{8}$, $y' = 0$.

i.e. $x \approx 0.94\ (54°)$ and $\approx 2.57\ (147°)$.

$$y'' = -\sin x - 4\sin 2x = -\sin x(1 + 8\cos x);$$

when $x \approx 0.94$, $y'' < 0 \Rightarrow$ loc. max;

$x \approx 2.57$, $y'' > 0 \Rightarrow$ loc. min;

$y'' = 0 \iff x = 0$, or $x = \pi$, or $\underbrace{1 + 8\cos x = 0}$

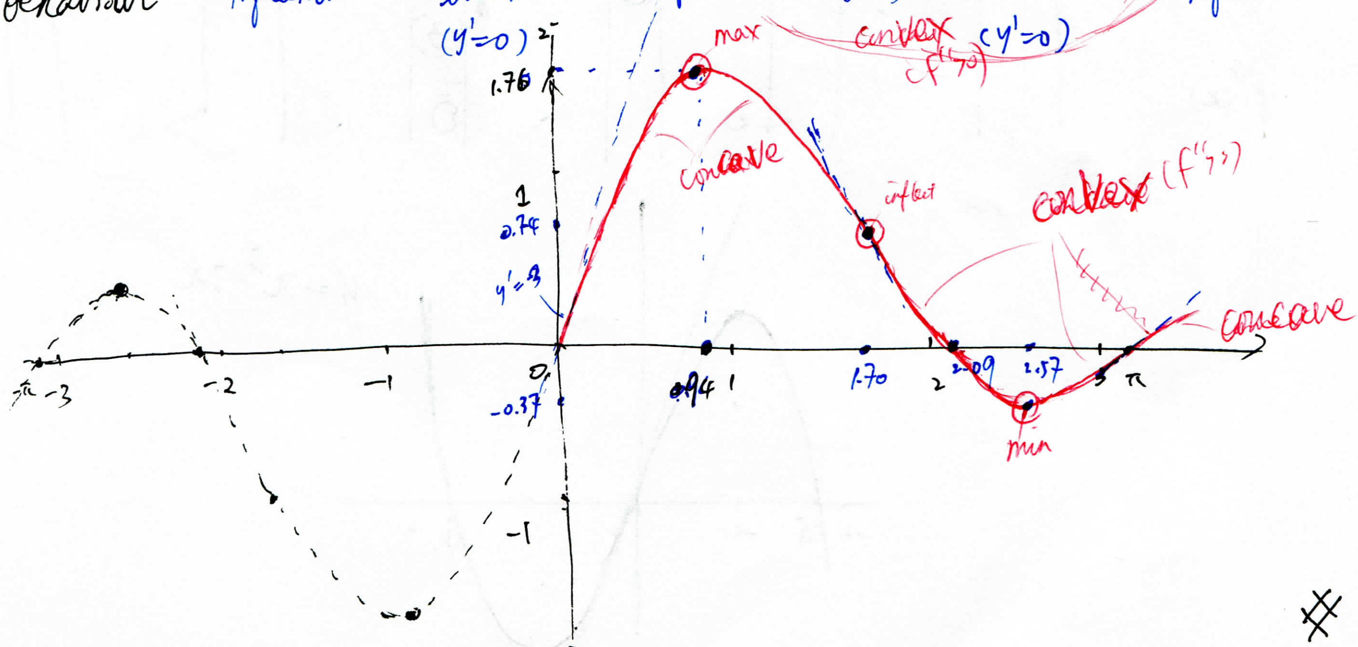
$\Updownarrow$

inflection point. $x \approx 1.70\ (97°)$.

List

x	0	0.94	1.70	2.09	2.57	3.14
y	0	1.76	0.74	0	−0.37	0
Behaviour	Inflection	loc. max ($y'=0$)	Inflection	($y'<0$)	loc. min ($y'=0$)	Inflection

concave ($f''<0$) convex ($f''>0$) concave



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Example 3 (Fikhtengol's [136] 1), [49] 3))

$$y = (x+2)^2 (x-1)^3 \; ;$$

- $y' = 2(x+2)(x-1)^3 + 3(x+2)^2(x-1)^2 = (x+2)(x-1)^2(5x+4)$

  $y'=0 \iff$ stationary point $\quad x_1 = -2, \quad x_2 = -\frac{4}{5}, \quad x_3 = 1 \, ;$

- $y'' \overset{\text{Leibniz rule}}{=} 2(x-1)(10x^2+16x+1)$.

  $y''(x_1=-2) = \cdots < 0$, $\quad y''(x_2 = -\frac{4}{5}) = \cdots > 0$, $\quad \underbrace{y''(x_3) = 0}_{\text{don't know yet!}}$

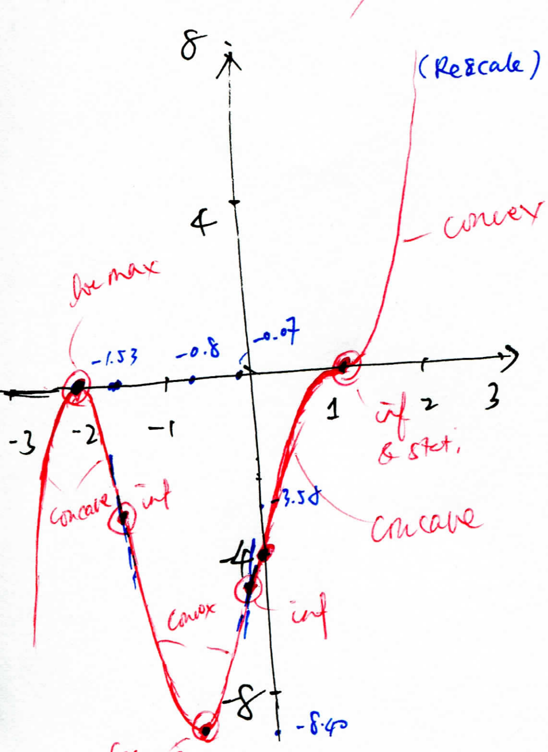
  $\Updownarrow$ loc. max $\qquad \Updownarrow$ loc. min

  $y''=0 \iff x = 1, \; -0.07, \; -1.53 \; ;$

List above information

x	-3	-2	-1.53	-0.8	-0.07	0	1	2
y	-64	0	-3.58	-8.40	-4.56	-4	0	16
Behaviour		loc. max ($y'=0$, $y''<0$)	Inflection	loc. min ($y'=0$) $y''>0$	Inflection		Inflection & $y'=0$.	

concave convex concave concave



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Tutorial 5

Topics: Quotient rule & Chain rule.

Questions: Evaluate the first derivatives of the function:

Quotient rule: 1a) $\frac{\sin(x)}{e^x}$ 1b) $\frac{1}{3x^2+2x+1}$ 1c) $\frac{\ln x}{\sqrt{x}}$

Chain rule: 2a) $\sin(x^6)$ 2b) $2^{(x^2)}$

2c) $\frac{1}{3}(\sqrt{x}+1)^3 + \frac{1}{2}(\sqrt{x}+1)^2 + (\sqrt{x}+1)$

2d) $e^{(\ln x)^3}$

Recall:

Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable functions.

- Quotient rule: $\left(\frac{f}{g}\right)' = \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

- Chain rule: $(f \circ g)' = \frac{d}{dx}\big(f(g(x))\big) = f'(g(x))\, g'(x)$.

<u>Sol$^{\underline{n}}$</u>

Q1 a)

$$\frac{d}{dx}\left(\frac{\sin x}{e^x}\right) = \frac{e^x \frac{d}{dx}\sin x - \sin x \frac{d}{dx}e^x}{(e^x)^2}$$

$$= \frac{e^x \cos x - e^x \sin x}{e^{2x}} = \frac{\cos x - \sin x}{e^x} \quad //$$

1b)

$$\frac{d}{dx}\left(\frac{1}{3x^2+2x+1}\right) = \frac{-\frac{d}{dx}(3x^2+2x+1)}{(3x^2+2x+1)^2}$$

$$= \frac{6x+2}{(3x^2+2x+1)^2} \quad //$$

Q1c)

$$\frac{d}{dx}\left(\frac{\ln(x)}{\sqrt{x}}\right) = \frac{\sqrt{x}\,\frac{d}{dx}\ln x - \ln x\,\frac{d}{dx}\sqrt{x}}{(\sqrt{x})^2} \quad , \text{ for } x > 0$$

$$= \frac{1}{x}\left(\frac{\sqrt{x}}{x} - \frac{\ln x}{2\sqrt{x}}\right) = \frac{2 - \ln x}{x\sqrt{x}}$$

2a) $$\frac{d}{dx}\sin(x^6) = \left(\frac{d}{du}\Big|_{u=x^6}\sin(u)\right)\left(\frac{d}{dx}x^6\right) = 6x^5\sin(x^6) \quad //$$

b) $$\frac{d}{dx}2^{(x^2)} = \frac{d}{dx}\left(e^{x^2\ln 2}\right) = \left(\frac{d}{du}\Big|_{u=x^2\ln 2} e^u\right)\left(\frac{d}{dx}x^2\ln 2\right)$$

$$= \left(e^{x^2\ln 2}\right)\left(2x\ln 2\right) = 2^{x^2}\ln 2^{2x} \quad //$$

2c) $\frac{d}{dx}\left(\frac{1}{3}(\sqrt{x}+1)^3 + \frac{1}{2}(\sqrt{x}+1)^2 + \sqrt{x}+1 \right)$

$$= \left[\frac{d}{du}\Big|_{u=\sqrt{x}+1} \left(\frac{u^3}{3} + \frac{u^2}{2} + u \right) \right] \left[\frac{d}{dx}(\sqrt{x}+1) \right]$$

$$= \left((\sqrt{x}+1)^2 + (\sqrt{x}+1) + 1 \right)\left(\frac{1}{2\sqrt{x}} \right)$$

$$= \frac{\sqrt{x}}{2} + \frac{3}{2} + \frac{3}{2\sqrt{x}} \quad //$$

2d)

$$\frac{d}{dx} e^{(\ln x)^3} = \left(\left.\frac{d}{du}\right|_{u=(\ln x)^3} e^u \right)\left(\frac{d}{dx} (\ln x)^3 \right)$$

$$= \left(\left.\frac{d}{du}\right|_{u=(\ln x)^3} e^u \right)\left(\left.\frac{d}{dv}\right|_{v=\ln x} v^3 \right)\left(\frac{d}{dx} \ln x \right)$$

$$= \left(e^{(\ln x)^3} \right)\left(3 (\ln x)^2 \right)\left(\frac{1}{x} \right)$$

$$= \frac{3}{x} (\ln x)^2 e^{(\ln x)^3}$$

//